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VIBRATION-PROOF CONDUIT FASTENING

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It is shown that the problem of determining the type and parameters of conduit end fastening from the eigenfrequency spectrum has a dual solution. A method of solving this problem is developed. Some examples are given.

Key words: *vibration-proof fastening, conduit, spectral problem, inverse problem.*

Introduction. Conduits are important elements of the fuel systems of cars, tractors, ships, planes, etc. Their vibrations often cause drumming, leading to discomfort for crew members and passengers. This is due to the fact that the frequency spectra of conduit vibrations are sometimes in a range hazardous to human health. To change the conduit vibration frequencies, it is not always reasonable to change the conduit length or attach concentrated masses. Therefore, to produce comfort conditions for passengers, it is required to determine the types of conduit fastening that provide the necessary (safe) range of conduit vibration frequencies. This refers not only to the fundamental vibration mode but also overtones. This problem is related to issues of noise suppression [1–3], acoustic diagnostics, [4–9] and the theory of inverse problems of mathematical physics [10, 11].

The goal of the present work was to determine the fastening parameters of a conduit filled with a fluid from the eigenfrequencies of its flexural vibrations. The problems of diagnosing the fastening of strings, membranes, and plates have been studied previously [12–19]. For conduits, however, the problem formulated here is apparently considered for the first time. In addition, unlike in [12–19], in the present work, four rather than two boundary conditions are sought, which significantly complicates the problem and requires the use of different methods for its solution.

Problems of calculating the eigenfrequencies of flexural vibrations of conduits were investigated in [20, 21]. However, the inverse problem — determining the boundary conditions from eigenfrequencies — was not studied in these papers. In addition, in [20, 21], only approximate methods (for example, the Galerkin and Rayleigh–Ritz methods) were considered, which are unsuitable for the solution of the problem formulated.

1. Primal Problem. The small free vibrations of a conduit filled with a fluid (which is incompressible) is described by the following equation [20] (see also [21, pp. 193–196]):

$$EI\,\frac{\partial^4 w}{\partial x^4}+(m+\bar m)\,\frac{\partial^2 w}{\partial t^2}+\bar m\,\frac{p_0}{\rho_0}\,\frac{\partial^2 w}{\partial x^2}=0.$$

Here $I=(\pi/4)(r^4-r_1^4)$ is the moment of inertia of the conduit cross section, EI is the rigidity of the conduit cross section, p_0 is the critical internal pressure, $m=\pi(r^2-r_1^2)\rho$ and $\bar m=\pi r_1^2\rho_0$ are the masses of the conduit and fluid per unit length l of the conduit, respectively, r and r_1 are the outer and inner radii of the cross section, respectively, ρ is the density of the conduit material, and ρ_0 is the fluid density.

Introducing the dimensionless variables $\tilde x=x/l$, $\tilde w=w/r$, and $\tilde t=t/\tau$ and representing the deflection in the form $\tilde w(\tilde x,\tilde t)=X(\tilde x)\,e^{i\omega\tilde t}$, we reduce the initial equation to the ordinary linear differential equation of fourth order with constant coefficients

$$X^{(4)}+aX''-\omega^2X=0, \qquad (1)$$

where $a=\bar m l^2p_0/(EI\rho_0)$.

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The linearly independent solutions of Eq. (1) are the functions $X_j = X_j(\tilde{x}, \omega) = \mathrm{e}^{\lambda_j \tilde{x}}$ $(j = 1, 2, 3, 4)$, where $\lambda_j = \lambda_j(\omega)$ are the various roots of the corresponding characteristic equation.

To formulate the spectral problem of free vibrations of the conduit, it is also necessary to specify boundary conditions. Since we consider the flexural vibrations of the conduit, the boundary conditions of the problem are similar to the boundary conditions of the problem of the free flexural vibrations of a bar. Generally, considering restraint, free support, free end, floating restraint, and various types of elastic restraint, the boundary conditions have the following form [14, 22]:

$$U_1(X) = a_1 X(0) + a_4 X'''(0) = 0, \qquad U_2(X) = a_2 X'(0) + a_3 X''(0) = 0; \tag{2}$$

$$U_3(X) = b_1 X(1) + b_4 X'''(1) = 0, \qquad U_4(X) = b_2 X'(1) + b_3 X''(1) = 0. \tag{3}$$

The equation for the frequencies is obtained from the condition that the characteristic determinant is equal to zero [22]:

$$\Delta(\omega) = \begin{vmatrix} U_1(X_1) & U_1(X_2) & U_1(X_3) & U_1(X_4) \\ U_2(X_1) & U_2(X_2) & U_2(X_3) & U_2(X_4) \\ U_3(X_1) & U_3(X_2) & U_3(X_3) & U_3(X_4) \\ U_4(X_1) & U_4(X_2) & U_4(X_3) & U_4(X_4) \end{vmatrix}. \tag{4}$$

Some approximate methods for calculating the roots of the characteristic determinant are described in [23].

2. Inverse Problem. The matrix composed of the coefficients a_j of the forms $U_1(X_k)$ and $U_2(X_k)$ will be denoted by A, and the matrix composed of the coefficients b_j of the forms $U_3(X_k)$ and $U_4(X_k)$ by B:

$$A = \left\| \begin{matrix} a_1 & 0 & 0 & a_4 \\ 0 & a_2 & a_3 & 0 \end{matrix} \right\|, \qquad B = \left\| \begin{matrix} b_1 & 0 & 0 & b_4 \\ 0 & b_2 & b_3 & 0 \end{matrix} \right\|.$$

The second-order minors formed from the ith and jth columns of the matrices A and B will be denoted by A_{ij} and B_{ij}, respectively.

We note that determining the boundary conditions does not mean finding all coefficients a_j and b_j since, for example, the boundary conditions $X(0) = 0$ and $X'(0) = 0$ are equivalent to $X(0) - X'(0) = 0$ and $X(0) + X'(0) = 0$ but their coefficients a_j are different.

The present study seeks not only to exactly identify all coefficients a_j and b_j but also to determine the boundary conditions, which is equivalent to finding the linear shells $\langle \boldsymbol{a}_1, \boldsymbol{a}_2 \rangle$ and $\langle \boldsymbol{b}_1, \boldsymbol{b}_2 \rangle$ constructed from the vectors

$$\boldsymbol{a}_1 = (a_1, 0, 0, a_4)^{\mathrm{t}}, \quad \boldsymbol{a}_2 = (0, a_2, a_3, 0)^{\mathrm{t}}, \quad \boldsymbol{b}_1 = (b_1, 0, 0, b_4)^{\mathrm{t}}, \quad \boldsymbol{b}_2 = (0, b_2, b_3, 0)^{\mathrm{t}}.$$

In terms of problem (1)–(3), the inverse problem of finding boundary conditions (2) and (3) can be formulated as follows: the coefficients a_j and b_j of the form $U_i(X_m)$ $(i, j, m = 1, 2, 3, 4)$ of problem (1)–(3) are unknown. The ranks of the matrices A and B composed of these coefficients are equal to two. The eigenvalues ω_k of problem (1)–(3) are unknown. It is required to find the linear shells $\langle \boldsymbol{a}_1, \boldsymbol{a}_2 \rangle$ and $\langle \boldsymbol{b}_1, \boldsymbol{b}_2 \rangle$.

3. Duality of the Solution of the Inverse Problem. To simplify the calculations, it is necessary to introduce a new notation. We denote by C a matrix of dimension 4×8:

$$C = \left\| \begin{matrix} A & 0 \\ 0 & B \end{matrix} \right\|. \tag{5}$$

The elements of the matrix C will be denoted by c_{ij}, and the minors of the matrix C composed of columns with the numbers k_1, k_2, k_3, and k_4, by $M_{k_1 k_2 k_3 k_4}$:

$$M_{k_1 k_2 k_3 k_4} = \begin{vmatrix} c_{1k_1} & c_{1k_2} & c_{1k_3} & c_{1k_4} \\ c_{2k_1} & c_{2k_2} & c_{2k_3} & c_{2k_4} \\ c_{3k_1} & c_{3k_2} & c_{3k_3} & c_{3k_4} \\ c_{4k_1} & c_{4k_2} & c_{4k_3} & c_{4k_4} \end{vmatrix}.$$

In the new notation, boundary conditions (2) and (3) can be written as

$$U_i(X_m) = \sum_{j=1}^{4} \Big[c_{ij} X_m^{(j-1)}(0) + c_{i,4+j} X_m^{(j-1)}(1) \Big], \qquad i = 1, 2, 3, 4. \tag{6}$$

In this notation, the inverse problem is formulated as follows: the coefficients c_{ij} of problem (1), (6) are unknown; the rank of the matrix C composed of these coefficients is equal to four; the minors A_{14}, A_{23}, B_{14}, and B_{23} of the matrices A and B constituting the matrix C are equal to zero; the eigenvalues ω_k of problem (1), (6) are known. It is required to find the linear shell $\langle \boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3, \boldsymbol{c}_4 \rangle$ constructed from the vectors $\boldsymbol{c}_i = (c_{i1}, c_{i2}, c_{i3}, c_{i4}, c_{i5}, c_{i6}, c_{i7}, c_{i8})^{\mathrm{t}}$ $(i = 1, 2, 3, 4)$.

We note that finding the linear shell $\langle \boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3, \boldsymbol{c}_4 \rangle$ is equivalent to finding the matrix C up to linear equivalence [24].

Let us show that the inverse problem has one or two solutions. Along with forms (6), we consider the following linear homogeneous forms:

$$\tilde{U}_i(X_m) = \sum_{j=1}^{4} \Big[\tilde{c}_{ij} X_m^{(j-1)}(0) + \tilde{c}_{i,4+j} X_m^{(j-1)}(1) \Big], \qquad i = 1, 2, 3, 4. \tag{7}$$

The matrix composed of the coefficients $\tilde{c}_{ij}$ will be denoted by $\tilde{C}$, its minors by $\tilde{M}_{k_1k_2k_3k_4}$, and the corresponding second-order minors by $\tilde{A}_{k_1k_2}$ and $\tilde{B}_{k_3-4,k_4-4}$. We introduce the following vectors:

$$\boldsymbol{c}_i^{+} = (\tilde{c}_{i1}, \tilde{c}_{i2}, \tilde{c}_{i3}, \tilde{c}_{i4}, \tilde{c}_{i5}, \tilde{c}_{i6}, \tilde{c}_{i7}, \tilde{c}_{i8})^{\mathrm{t}},$$
$$\boldsymbol{c}_i^{-} = (\tilde{c}_{i5}, \tilde{c}_{i6}, -\tilde{c}_{i7}, -\tilde{c}_{i8}, \tilde{c}_{i1}, \tilde{c}_{i2}, -\tilde{c}_{i3}, -\tilde{c}_{i4})^{\mathrm{t}}, \qquad i = 1, 2, 3, 4.$$

Theorem 1 (on the duality of the solution of the inverse problem)**.** *Let* $\operatorname{rank} C = \operatorname{rank} \tilde{C} = 4$*. If the eigenvalues* $\{\omega_k\}$ *of problem* (1), (6) *and the eigenvalues* $\{\tilde{\omega}_k\}$ *of problem* (1), (7) *coincide in view of their multiplicities, then* $\langle \boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3, \boldsymbol{c}_4 \rangle = \langle \boldsymbol{c}_1^{+}, \boldsymbol{c}_2^{+}, \boldsymbol{c}_3^{+}, \boldsymbol{c}_4^{+} \rangle$ *or* $\langle \boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3, \boldsymbol{c}_4 \rangle = \langle \boldsymbol{c}_1^{-}, \boldsymbol{c}_2^{-}, \boldsymbol{c}_3^{-}, \boldsymbol{c}_4^{-} \rangle$*.*

Proof. We note that the determinant (4) can be written as

$$\Delta(\omega_k) = \det(CD),$$

where

$$D = \begin{Vmatrix} X_1(0) & X_2(0) & X_3(0) & X_4(0) \\ X_1'(0) & X_2'(0) & X_3'(0) & X_4'(0) \\ X_1''(0) & X_2''(0) & X_3''(0) & X_4''(0) \\ X_1'''(0) & X_2'''(0) & X_3'''(0) & X_4'''(0) \\ X_1(1) & X_2(1) & X_3(1) & X_4(1) \\ X_1'(1) & X_2'(1) & X_3'(1) & X_4'(1) \\ X_1''(1) & X_2''(1) & X_3''(1) & X_4''(1) \\ X_1'''(1) & X_2'''(1) & X_3'''(1) & X_4'''(1) \end{Vmatrix}.$$

Using the Binet–Cauchy formula [25], we obtain

$$\Delta(\omega_k) = \sum_{1 \le k_1 < k_2 < \ldots < k_8 \le 8} M_{k_1k_2k_3k_4} f_{k_1k_2k_3k_4}, \tag{8}$$

where $f_{k_1k_2k_3k_4}$ are the fourth-order minors of the matrix D composed of the rows with the numbers k_1, k_2, k_3, and k_4.

Since $M_{k_1k_2k_3k_4} = 0$ for $k_3, k_4 \le 4$, $k_1, k_2 \ge 5$, then, using the Laplace theorem to calculate the determinant and taking into account that $\Delta(\omega_k) = 0$, we obtain

$$\sum_{\substack{1 \le k_1 < k_2 \le 4 \\ 5 \le k_3 < k_4 \le 8}} M_{k_1k_2k_3k_4} f_{k_1k_2k_3k_4}(\omega_k) = 0, \tag{9}$$

where

$$M_{k_1k_2k_3k_4} = A_{k_1,k_2}\, B_{k_3-4,k_4-4} \qquad (A_{14} = A_{23} = B_{14} = B_{23} = 0). \tag{10}$$

From the general theory of linear differential operators, it follows that the function $\Delta(\omega)$ is an integer function of order 1/2 (see [26]). From this it follows that the characteristic determinants $\Delta(\omega)$ and $\tilde{\Delta}(\omega)$ of problems (1), (6) and (1), (7) are linked by the relation

$$\Delta(\omega) \equiv K\tilde{\Delta}(\omega), \tag{11}$$

where K is a certain nonzero constant.

From (8) and (11), we obtain

$$\begin{aligned}
&[M_{1256} - K\tilde{M}_{1256}]f_{1256} + [M_{1257} - K\tilde{M}_{1257}]f_{1257} \\
&+ [M_{1268} - K\tilde{M}_{1268}]f_{1268} + [M_{1278} - K\tilde{M}_{1278}]f_{1278} \\
&+ [M_{1356} - K\tilde{M}_{1356}]f_{1356} + [M_{1357} - K\tilde{M}_{1357}]f_{1357} \\
&+ [M_{1368} - K\tilde{M}_{1368}]f_{1368} + [M_{1378} - K\tilde{M}_{1378}]f_{1378} \\
&+ [M_{2456} - K\tilde{M}_{2456}]f_{2456} + [M_{2457} - K\tilde{M}_{2457}]f_{2457} \\
&+ [M_{2468} - K\tilde{M}_{2468}]f_{2468} + [M_{2478} - K\tilde{M}_{2478}]f_{2478} \\
&+ [M_{3456} - K\tilde{M}_{3456}]f_{3456} + [M_{3457} - K\tilde{M}_{3457}]f_{3457} \\
&+ [M_{3468} - K\tilde{M}_{3468}]f_{3468} + [M_{3478} - K\tilde{M}_{3457}]f_{3478} \equiv 0.
\end{aligned} \tag{12}$$

It is easy to show the following: 1) $f_{1356} = -f_{1257}$, $f_{1257} = -f_{1356}$, $f_{1268} = -f_{2456}$, $f_{1278} = f_{3456}$, $f_{1368} = f_{2457}$, $f_{1378} = -f_{3457}$, and $f_{2478} = -f_{3468}$; 2) the functions $f_{1256}(\omega_m)$, $f_{1257}(\omega_m)$, $f_{1268}(\omega)$, $f_{1278}(\omega)$, $f_{1357}(\omega)$, $f_{1368}(\omega)$, $f_{1378}(\omega)$, $f_{2468}(\omega)$, $f_{2478}(\omega)$, and $f_{3478}(\omega)$ form a linearly independent system of functions. This implies the duality of the solution of the inverse problem. Indeed, the linear independence of the corresponding functions leads to the equalities

$$M_{1256} = K\tilde{M}_{1256}; \tag{13}$$

$$M_{1357} = K\tilde{M}_{1357}; \tag{14}$$

$$M_{2468} = K\tilde{M}_{2468}; \tag{15}$$

$$M_{3478} = K\tilde{M}_{3478}; \tag{16}$$

$$M_{1257} - M_{1356} = K(\tilde{M}_{1356} - \tilde{M}_{1257}); \tag{17}$$

$$M_{1268} - M_{2456} = K(\tilde{M}_{1268} - \tilde{M}_{2456}); \tag{18}$$

$$M_{1378} - M_{3457} = K(\tilde{M}_{1378} - \tilde{M}_{3457}); \tag{19}$$

$$M_{2478} - M_{3468} = K(\tilde{M}_{2478} - \tilde{M}_{3468}); \tag{20}$$

$$M_{1278} + M_{3456} = K(\tilde{M}_{1278} + \tilde{M}_{3456}); \tag{21}$$

$$M_{1368} + M_{2457} = K(\tilde{M}_{1368} + \tilde{M}_{2457}). \tag{22}$$

To prove the aforesaid, we consider five cases: 1) $M_{1256} \neq 0$; 2) $M_{1357} \neq 0$; 3) $M_{2468} \neq 0$; 4) $M_{3478} \neq 0$; 5) $M_{1256} = M_{1357} = M_{2468} = M_{3478} = 0$. Without loss of generality, we assume that $M_{1256} = K\tilde{M}_{1256} \neq 0$ (the first case occurs).

From the equalities $M_{1256} = A_{12}B_{12} = a_1a_2b_1b_2$ and $\tilde{M}_{1256} = \tilde{A}_{12}\tilde{B}_{12} = \tilde{a}_1\tilde{a}_2\tilde{b}_1\tilde{b}_2$, it follows that the elements a_1, a_2, b_1, b_2, $\tilde{a}_1$, $\tilde{a}_2$, $\tilde{b}_1$, and $\tilde{b}_2$ of the matrix C are different from zero.

We divide the 1st, 2nd, 3rd, and 4th rows of the matrix C by a_1, a_2, b_1, b_2, respectively, and the 1st, 2nd, 3rd, and 4th rows of the matrix $\tilde{C}$ by $\tilde{a}_1$, $\tilde{a}_2$, $\tilde{b}_1$, $\tilde{b}_2$, respectively. As a result of these transformations, the matrices C and $\tilde{C}$, up to linear equivalence, become

$$C=\begin{Vmatrix} 1 & 0 & 0 & a_4 & 0 & 0 & 0 & 0 \\ 0 & 1 & a_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & b_4 \\ 0 & 0 & 0 & 0 & 0 & 1 & b_3 & 0 \end{Vmatrix}, \qquad \tilde C=\begin{Vmatrix} 1 & 0 & 0 & \tilde a_4 & 0 & 0 & 0 & 0 \\ 0 & 1 & \tilde a_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \tilde b_4 \\ 0 & 0 & 0 & 0 & 0 & 1 & \tilde b_3 & 0 \end{Vmatrix}.$$

From this representation of the matrices C and $\tilde C$ and equality (13), it follows that $K = 1$.

Relations (14) and (17) imply that $a_3b_3 = \tilde a_3\tilde b_3$ and $b_3 - a_3 = \tilde b_3 - \tilde a_3$, whence, according to the Vieta theorem, we obtain the following alternative:

$$a_3 = \tilde a_3, \quad b_3 = \tilde b_3 \quad \text{or} \quad a_3 = -\tilde b_3, \quad b_3 = -\tilde a_3.$$

Similarly, from (15) and (18) it follows that

$$a_4 = \tilde a_4, \quad b_4 = \tilde b_4 \quad \text{or} \quad a_4 = -\tilde b_4, \quad b_4 = -\tilde a_4.$$

Thus, the first case $M_{1256} = K\tilde M_{1256} \neq 0$ is split into four cases:
1) $a_3 = \tilde a_3$, $b_3 = \tilde b_3$, $a_4 = \tilde a_4$, and $b_4 = \tilde b_4$;
2) $a_3 = -\tilde b_3$, $b_3 = -\tilde a_3$, $a_4 = -\tilde b_4$, and $b_4 = -\tilde a_4$;
3) $a_3 = -\tilde b_3$, $b_3 = -\tilde a_3$, $a_4 = \tilde a_4$, and $b_4 = \tilde b_4$;
4) $a_3 = \tilde a_3$, $b_3 = \tilde b_3$, $a_4 = -\tilde b_4$, and $b_4 = -\tilde a_4$.

Actually, however, two rather than four cases occur (cases 3 and 4 are particular cases of 1 and 2). Indeed, let case 3 occurs. Then, relations (21) or (22) leads to the equality

$$(a_3 + b_3)(a_4 + b_4) = 0. \tag{23}$$

From this we obtain $a_3 + b_3 = 0$ or $a_4 + b_4 = 0$. For $a_3 + b_3 = 0$, case 3 reduces to case 1 since $-\tilde b_3 = a_3 = -b_3 = \tilde a_3$ and, hence, $\tilde b_3 = b_3$, $\tilde a_3 = a_3$. For $a_4 + b_4 = 0$, case 3 reduces to case 2 since $\tilde a_4 = a_4 = -b_4 = -\tilde b_4$ and, hence, $\tilde b_4 = -a_4$, $\tilde a_4 = -b_4$. In contrast to (21) or (22), the remaining equalities (13)–(20) do not give new constraints. Thus, in case 3, case 1 or case 2 occur. In case 4, as in case 3, relation (21) or (22) leads to equality (23). For $a_3 + b_3 = 0$, case 4 reduces to case 2, and for $a_4 + b_4 = 0$, case 4 reduces to case 1. As a result, we obtain $\langle \boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3, \boldsymbol{c}_4\rangle = \langle \boldsymbol{c}_1^+, \boldsymbol{c}_2^+, \boldsymbol{c}_3^+, \boldsymbol{c}_4^+\rangle$ or $\langle \boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3, \boldsymbol{c}_4\rangle = \langle \boldsymbol{c}_1^-, \boldsymbol{c}_2^-, \boldsymbol{c}_3^-, \boldsymbol{c}_4^-\rangle$.

Thus, the solution of the inverse problem is dual. The theorem is proved.

We note that, for $\langle \boldsymbol{c}_1^+, \boldsymbol{c}_2^+, \boldsymbol{c}_3^+, \boldsymbol{c}_4^+\rangle = \langle \boldsymbol{c}_1^-, \boldsymbol{c}_2^-, \boldsymbol{c}_3^-, \boldsymbol{c}_4^-\rangle$, the two solutions (multiple solutions) coincide. This case occurs, for example, for the restraint–restraint fastening.

Thus, the problem of seeking the unknown boundary conditions from the eigenfrequencies of conduit flexural vibrations has two solutions. These solutions can be constructed using two methods based on the representation of the characteristic determinant in the form of the infinite product [14]:

$$\Delta(\omega) \equiv K \prod_{k=1}^{\infty} \Big(1 - \frac{\omega}{\omega_k}\Big).$$

These methods, however, proved ineffective because of the significant accumulation of errors in calculations of the corresponding infinite product. Therefore, in the present work, we used a different method based on solving a system of linear algebraic equations.

4. Method of Seeking the Boundary Conditions. Let ω_k be nine eigenfrequencies from the entire spectrum of problem (1), (6). Then, the equality $\Delta(\omega_k) = 0$ form a system of nine linear algebraic equations for ten unknowns $x_1, x_2, \ldots, x_n$:

$$\Delta(\omega_k) = x_1 f_{1257}(\omega_k) + x_2 f_{1268}(\omega_k) + x_3 f_{1368}(\omega_k) + x_4 f_{1278}(\omega_k) + x_5 f_{1378}(\omega_k)$$
$$+ x_6 f_{2478}(\omega_k) + x_7 f_{1357}(\omega_k) + x_8 f_{2468}(\omega_k) + x_9 f_{1256}(\omega_k) + x_{10} f_{3478}(\omega_k) = 0. \tag{24}$$

Here

$$\begin{gathered} x_1 = M_{1257} - M_{1356}, \quad x_2 = M_{1268} - M_{2456}, \quad x_3 = M_{1368} + M_{2457}, \\ x_4 = M_{1278} + M_{3456}, \quad x_5 = M_{1378} - M_{3457}, \quad x_6 = M_{2478} - M_{3468}, \\ x_7 = M_{1357}, \quad x_8 = M_{2468}, \quad x_9 = M_{1256}, \quad x_{10} = M_{3478}. \end{gathered} \tag{25}$$

If rank $\|f_{k_1k_2k_3k_4}(\omega_k)\| = 9$ (this matrix has dimension 10×9), the system of linear algebraic equations (24) has a unique (up to a constant factor) solution $x_1, x_2, \dots, x_{10}$. The two required matrices C are found (up to equivalence) from the values of $x_1, x_2, \dots, x_{10}$. Some examples are given below.

Example 1. *Free support–restraint.* We consider the differential equation

$$X^{(4)} + 2X'' - \omega^2 X = 0. \tag{26}$$

Let nine eigenfrequencies ω_k problems (26), (2), (3) be known: $\omega_1 = 14.65$, $\omega_2 = 49.10$, $\omega_3 = 103.34$, $\omega_4 = 177.34$, $\omega_5 = 271.09$, $\omega_6 = 384.58$, $\omega_7 = 670.79$, $\omega_8 = 843.504$, and $\omega_9 = 1035.96$. We find their corresponding boundary conditions. Computer calculations yield the following solution of system (24): $x_1 = K$, $x_i = 0$, $i = 2, 3, \dots, 10$. Here and below, $K = \text{const} \neq 0$ (the data are given to two decimal places; real calculations were performed on a computer with 40 decimal places).

From the equality $x_1 = M_{1257} - M_{1356} = K$, it follows that $M_{1257} \neq 0$ or $M_{1356} \neq 0$ (otherwise, the ranks of the matrices A and B are equal to zero, which contradicts the fact that their ranks are equal to two). We find the matrices C that correspond to these cases.

1. Let $M_{1257} \neq 0$. Then, $a_1 \neq 0$, $a_2 \neq 0$, $b_1 \neq 0$, and $b_3 \neq 0$. From this, we find that the matrix C (up to linear equivalence) has the form

$$C = \begin{Vmatrix} 1 & 0 & 0 & a_4 & 0 & 0 & 0 & 0 \\ 0 & 1 & a_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & b_4 \\ 0 & 0 & 0 & 0 & 0 & b_2 & 1 & 0 \end{Vmatrix}.$$

From the equalities $M_{1357} = 0$ and $M_{1256} = 0$, we obtain $a_3 = 0$ and $b_2 = 0$, and from the equalities $x_3 = M_{1368} + M_{2457} = 0$ and $x_4 = M_{1278} + M_{3456} = 0$, we have $a_4 = 0$ and $b_4 = 0$.

Hence, the matrix C has the form

$$C = \begin{Vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{Vmatrix}.$$

2. Let $M_{1356} \neq 0$. Then, using the same method as in the case $M_{1257} \neq 0$, we obtain

$$C = \begin{Vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{Vmatrix}.$$

Thus, according to the theorem, we obtain two solutions:

1) restraint–free support:

$$X(0) = 0, \quad X'(0) = 0, \quad X(1) = 0, \quad X''(1) = 0;$$

2) free support–restraint:

$$X(0) = 0, \quad X''(0) = 0, \quad X(1) = 0, \quad X'(1) = 0.$$

Example 2. *Elastic fastening.* As in example 1, we consider the differential equation (26). Let nine eigenfrequencies ω_k of problem (26), (2), (3) be known: $\omega_1 = 21.67$, $\omega_2 = 60.87$, $\omega_3 = 120.06$, $\omega_4 = 198.98$, $\omega_5 = 297.66$, $\omega_6 = 416.08$, $\omega_7 = 712.15$, $\omega_8 = 889.79$, and $\omega_9 = 1087.18$. In this case, the rank of system (24) is equal to nine, and the nonzero components of the solution are the following:

$$x_5 = -3K, \qquad x_7 = K, \qquad x_{10} = -2K. \tag{27}$$

From the equality $x_7 = M_{1357} \neq 0$, we obtain $a_1 \neq 0$, $a_3 \neq 0$, $b_1 \neq 0$, and $b_3 \neq 0$, whence it follows that the matrix C (up to linear equivalence) has the form

$$C = \begin{Vmatrix} 1 & 0 & 0 & a_4 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & b_4 \\ 0 & 0 & 0 & 0 & 0 & b_2 & 1 & 0 \end{Vmatrix} \qquad (K = 1).$$

From this and from (25) and (27), we have $a_2 = 0$, $b_2 = 0$, $a_4 - b_4 = -3$, and $a_4 b_4 = -2$. Hence, $a_2 = 0$, $b_2 = 0$, $a_4 = -1$, and $b_4 = 2$ or $a_2 = 0$, $b_2 = 0$, $a_4 = -2$, and $b_4 = 1$.

Thus, the matrix C (up to linear equivalence) has the form

$$C = \begin{Vmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{Vmatrix}$$

or

$$C = \begin{Vmatrix} 1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{Vmatrix}.$$

As a result, we obtain the following boundary conditions:

$$X(0) - X'''(0) = 0, \quad X''(0) = 0, \quad X(1) + 2\,X'''(1) = 0, \quad X''(1) = 0$$

or

$$X(0) - 2\,X'''(0) = 0, \quad X''(0) = 0, \quad X(1) + X'''(1) = 0, \quad X''(1) = 0.$$

Conclusions. The study showed the duality of the solution of the problem of determining the type of conduit end fastening from the eigenfrequency spectrum. A method for solving this problem from nine eigenfrequencies was developed. Some examples were given.

The duality of the solution of the problem can be explained as follows. If fluid does not flow in a conduit, its ends are equivalent, and the type of conduit end fastening is determined with accuracy up to rearrangement. For example, if the left end of the conduit is fixed by a spring with relative flexural rigidity equal to unity, and the right end is fixed by a spring with relative flexural rigidity equal to two, only a conduit whose left end is fixed by a spring with relative flexural rigidity equal to two and whose right end is fixed by a spring with relative flexural rigidity equal to unity has the same frequency spectrum.

The results obtained can be useful in choosing the type of fastening such that the conduit vibrations have the required (safe) frequency spectrum. In addition, these results are applicable for acoustic diagnostics of the conduit fastening (in this case, the devices measuring eigenfrequencies should have very high accuracy).

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